Proceedings of the Jangjeon Mathematical Society
27 (2024), No. 2, pp. 145 – 156

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2024.27.2.145

ON S-ARMENDARIZ RINGS

MOHAMED CHHITI AND SALAH EDDINE MAHDOU

ABSTRACT. In this paper, we introduce a new class of rings called S-Armendariz rings, which is a weak version of Armendariz rings. Any Armendariz ring is naturally an S-Armendariz ring, and when $S \subseteq U(R)$, these two classes coincide. We study the transfer of this notion to various contexts of commutative ring extensions such as direct product, trivial ring extensions and amalgamated algebras along ideals. Our results generate new families of examples of non Armendariz S-Armendariz rings.

2010 MATHEMATICS SUBJECT CLASSIFICATION. 13A15, 13B99, 13E15.

KEYWORDS AND PHRASES. S-Armendariz ring, trivial ring extension, amalgamated algebras along an ideal.

1. INTRODUCTION

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring, and let $Reg(R)$, $Z(R)$, $U(R)$ and $R[X]$ denote the set of all regular elements of R, the set of all zero-divisors of R, the group of units of a ring R, and the polynomial ring over R, respectively. By a "local" ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal.

Let R be a commutative ring. The content $C(f(X))$ of a polynomial $f(X) \in R[X]$ is the ideal of R generated by all coefficients of $f(X)$. One of its properties is that $C(.)$ is semi-multiplicative, that is $C(f(X)g(X)) \subseteq C(f(X))C(g(X))$; and a polynomial $f(X) \in R[X]$ is said to be Gaussian over R if $C(f(X)g(X)) = C(f(X))C(g(X))$ for every polynomial $g(X) \in R[X]$. A polynomial $f(X) \in R[X]$ is Gaussian provided $C(f(X))$ is locally principal by [28, Remark 1.1]. A ring R is called a Gaussian ring if $C(f(X)g(X)) = C(f(X))C(g(X))$ for any polynomials $f(X), g(X)$ with coefficients in R. A domain is Gaussian if and only if it is a Prüfer domain. See for instance [3, 7, 8, 25, 28].

In [40], Rege and Chhawchharia introduced the notion of an Armendariz ring as a ring R such that, for all polynomials $f(X) = \sum_{i=0}^{m} a_iX^i$ and $g(X) = \sum_{i=0}^{n} b_iX^i \in R[X]$ satisfying $f(X)g(X) = 0$, we have $a_ib_j = 0$ for all i and j (that is $C(f(X))C(g(X)) = 0$). It is easy to see that subrings of Armendariz rings are also Armendariz. E. Armendariz ([5, Lemma 1]) noted that any reduced ring (i.e., a ring without non-zero nilpotent elements) is an Armendariz ring. Also, D.D. Anderson and V. Camillo ([3]) showed that a ring R is Gaussian if and only if every homomorphic image of R is Armendariz. See for instance [3, 4, 5, 6, 8, 22, 28, 32, 35, 40].

In [1], Anderson and Dumitrescu introduced the concept of S-finite modules, where S is a multiplicatively closed subset, as follows: an R-module E is called an S-finite module if there exist a finitely generated R-submodule L of E and $s \in S$ such that $sE \subseteq L$. Also, they introduced the concept of S-Noetherian rings as follows : a ring R is called S-Noetherian if every ideal of R is S-finite. Recently, in [9], Bennis and El Hajoui investigated the S-versions of finitely presented modules and coherent modules which are called, respectively, S-finitely presented modules and S-coherent modules. An R-module M is called an S-finitely presented module for some multiplicatively closed subset S of R if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$, where F is a finitely generated free R-module and K is an S-finite R-module. Moreover, an R-module M is said to be S-coherent if it is finitely generated and every finitely generated submodule of M is S-finitely presented. They showed that the S-coherent rings have a characterization similar to the classical one given by Chase for coherent rings (see [9, Theorem 3.8]). Any coherent ring is S-coherent and any S-noetherian ring is S-coherent. Suitable background on S-Noetherian property is [1, 9, 10, 26, 27, 33, 34, 41].

Let R be a ring and E be an R-module. Then $R \propto E$, the trivial ring extension of R by E, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by $(r_1, e_1)(r_2, e_2) := (r_1r_2, r_1e_2 + r_2e_1)$ for all $r_1, r_2 \in R$ and all $e_1, e_2 \in E$. The basic properties of trivial ring extensions are summarized in the books [23, 29]. For the reader's convenience, recall that if I is an ideal of A and E' is a submodule of E such that $IE \subseteq E'$, then $J := I \propto E'$ is an ideal of R; ideals of R need not be of this form [31, Example 2.5]. However, prime (resp., maximal) ideals of R have the form $P \propto E$, where P is a prime (resp., maximal) ideal of A [29, Theorem 25.1(3)]. If (A, M) is a local ring with maximal ideal M and E an A-module with $ME = 0$, then $R := A \propto E$ is local total ring of fractions from [31, Proof of Theorem 2.6]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. Suitable background on commutative trivial ring extensions is [2, 7, 17, 18, 19, 23, 29, 30, 31, 36].

Let A and B be two rings with identity, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of* A *and* B *along* J *with respect to* f. This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [14, 15, 16]). The interest of amalgamation resides, partly, in its ability to cover several basic constructions in commutative algebra, including pullbacks and trivial ring extensions (also called Nagata's idealizations) (cf. [39, page 2]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([12, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [11]) are strictly related to it ([12, Example 2.7 and Remark 2.8]). In [12], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. Moreover, in [15], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, chain properties and Krull dimension. See for instance [12, 13, 14, 15, 16, 18, 20, 21, 37, 38].

In this paper, we introduce a new class of rings called S-Armendariz rings. Any Armendariz ring is naturally an S-Armendariz ring, and when $S \subseteq U(R)$, these two classes coincide. We study the transfer of this notion to various context of commutative ring extensions such as direct product, trivial ring extensions and amalgamated algebras along ideals. Our results generate new families of examples of non Armendariz S-Armendariz rings.

2. Main Results

Definition 2.1. *Let R be a ring and S be a multiplicatively closed subset of R. Then R is called S-Armendariz if, for all polynomials $f(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{j=0}^{n} b_j X^j$ in $R[X]$ such that $f(X)g(X) = 0$, then for all i, j, there exists $s \in S$ such that $sa_ib_j = 0$.*

Any Armendariz ring is naturally S-Armendariz, and if $S \subseteq U(R)$, these two classes coincide.

Now, we give an example of a non-Armendariz S-Armendariz ring.

Example 2.2. *Let A be an Armendariz ring, B be a non-Armendariz ring and set $S := \{(1, 1), (1, 0)\}$, which is a multiplicatively closed subset of $A \times B$. Then:*

(1) *$A \times B$ is an S-Armendariz ring.*
(2) *$A \times B$ is not an Armendariz ring.*

Proof.

(1) Let $f(X) = \sum_{i=0}^{m}(a_i, e_i)X^i$ and $g(X) = \sum_{j=0}^{n}(b_j, f_j)X^j$ in $(A \times B)[X]$ such that $f(X)g(X) = 0$, where m and n are positive integers. Set

$f_1(X) = \sum_{i=0}^{m} a_i X^i \in A[X]$, $f_2(X) = \sum_{i=0}^{m} e_i X^i \in B[X]$, $g_1(X) = \sum_{j=0}^{n} b_j X^j \in A[X]$ and $g_2(X) = \sum_{j=0}^{n} f_j X^j \in B[X]$. We have

$$\begin{aligned} 0 &= f(X)g(X) \\ &= (f_1(X)g_1(X), g_2(X)f_2(X)) \end{aligned}$$

which imply that $f_1(X)g_1(X) = 0$. Hence, $a_ib_j = 0$ for each i, j since A is Armendariz. Therefore, $(1, 0)(a_i, e_i)(b_j, f_j) = (0, 0)$ which means that $A \times B$ is an S-Armendariz ring since $(1, 0) \in S$.

(2) $A \times B$ is not an Armendariz ring by [6, Proof of Theorem 2.6] since B is not an Armendariz ring, as desired. □

Now, we study the transfer of S-Armendariz property to a direct product of rings.

Theorem 2.3. *Let S_i be a multiplicatively closed subset of a ring R_i for each $i = 1, ..., n$, set $R := \prod_{i=1}^{n} R_i$ and $S := \prod_{i=1}^{n} S_i$, which is a multiplicatively closed subset of R. Then R is S-Armendariz if and only if R_i is S_i-Armendariz for each $i = 1, ..., n$.*

Proof. By induction, it suffices to show the proof for $n = 2$. Assume that $R_1 \times R_2$ is an $(S_1 \times S_2)$-Armendariz ring. We show that R_1 is an S_1-Armendariz ring (it is the same for R_2).

Let $f(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{i=0}^{n} b_i X^i$ be two polynomials in $R_1[X]$ such that $f(X)g(X) = 0$, where m and n are positive integers. Set $f_1(X) = \sum_{i=0}^{m} (a_i, 0) X^i$ and $g_1(X) = \sum_{i=0}^{n} (b_i, 0) X^i$ $(\in (R_1 \times R_2)[X])$. We have

$$\begin{aligned} f_1(X)g_1(X) &= (f(X)g(X), 0) \\ &= (0, 0). \end{aligned}$$

Hence, for each i, j, there exists $(s_1, s_2) \in S_1 \times S_2$ such that $(s_1, s_2)(a_i, 0)(b_j, 0) = (0, 0)$ since $R_1 \times R_2$ is an $(S_1 \times S_2)$-Armendariz ring. Therefore, $s_1 a_i b_j = 0$ and this shows that R_1 is an S_1-Armendariz ring.

Conversely, assume that R_i is an S_i-Armendariz rings for $i = 1, 2$. Let $f(X) = \sum_{i=0}^{m} (a_i, e_i) X^i$ and $g(X) = \sum_{j=0}^{n} (b_j, f_j) X^j$ be two polynomials in $(R_1 \times R_2)[X]$ such that $f(X)g(X) = 0$, where m and n are positive integers. Set $f_1(X) = \sum_{i=0}^{m} a_i X^i \in R_1[X]$, $f_2(X) = \sum_{i=0}^{m} e_i X^i \in R_2[X]$, $g_1(X) = \sum_{j=0}^{n} b_j X^j \in$

$R_1[X]$ and $g_2(X) = \sum_{j=0}^{n} f_j X^j \in R_2[X]$. We have

$$\begin{aligned} 0 &= f(X)g(X) \\ &= (f_1(X)g_1(X), f_2(X)g_2(X)) \end{aligned}$$

which implies that $f_1(X)g_1(X) = 0$ and $f_2(X)g_2(X) = 0$. Hence, for each i, j, there exists $s_1 \in S_1$ and $s_2 \in S_2$ such that $s_1 a_i b_j = 0$ and $s_2 e_i f_j = 0$ since R_k is an S_k-Armendariz rings for $k = 1, 2$. Hence, $(s_1, s_2)(a_i, e_i)(b_j, f_j) = (0, 0)$ and this completes the proof of Theorem 2.3. □

Using Theorem 2.3 in the case when $S_i \subseteq U(R_i)$ for each $i = 1, \ldots n$, we regain the result [6, Theorem 2.6].

Corollary 2.4. *For $i = 1, ..., n$, let R_i be a ring and set $R := \prod_{i=1}^{n} R_i$. Then R is Armendariz if and only if R_i is Armendariz for each $i = 1, ..., n$.*

Next, we study the transfer of S-Armendariz property to localization.

Theorem 2.5. *Let R be a ring and S be a multiplicatively closed subset of R. Then :*

(1) *If R is an S-Armendariz ring, then $S^{-1}R$ is an Armendariz ring.*
(2) *If R is an S-Armendariz ring, then R_M is an Armendariz ring for each $M \in Max(R)$ such that $S \subseteq R - M$.*
(3) *Let $M \in Max(R)$ and set $S := R - M$. Then, R is an S-Armendariz ring if and only if R_M is an Armendariz ring.*

Proof.

(1) Assume that R is an S-Armendariz ring and let $f(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{j=0}^{n} b_j X^j$ be two polynomials in $R[X]$ such that $(f(X)/1)(g(X)/1) = 0$ in $S^{-1}R[X]$. Hence, there exists $s_1 \in S$ such that $s_1 f(X)g(X) (= (s_1 f(X))g(X) = 0$. Therefore, for each i, j, there exists $s_2 \in S$ such that $s_2(s_1 a_i)b_j (= (s_2 s_1)a_i b_j) = 0$ (since R is an S-Armendariz ring and $s_1 f(X), g(X) \in R[X]$ such that $(s_1 f(X))g(X) = 0$). Hence, $(a_i/1)(b_j/1) = 0$ in $S^{-1}R$, which means that $S^{-1}R$ is an Armendariz ring.
(2) Assume that R is an S-Armendariz ring and let $f(X) = \sum_{i=0}^{m} a_i X^i$ and $g(X) = \sum_{j=0}^{n} b_j X^j$ be two polynomials in $R[X]$ such that $(f(X)/1)(g(X)/1) = 0$ in $R_M[X]$. Hence, there exists $c \in R - M$

such that $cf(X)g(X)(= (cf(X))g(X)) = 0$. Therefore, for each i, j, there exists $s \in S$ such that $s(ca_i)b_j(= (sc)a_ib_j) = 0$ since R is an S-Armendariz ring. But, $sc \in R - M$ since $c \in R - M$ and $s \in S \subseteq R - M$. Hence, $(a_i/1)(b_j/1) = 0$ in R_M, as desired.

(3) Let $M \in Max(R)$ and set $S := R - M$. If R is an S-Armendariz ring, then $R_M(= S^{-1}R)$ is an Armendariz ring by (1). Conversely, assume that $R_M(= S^{-1}R)$ is an Armendariz ring and let $f(X) = \sum_{i=0}^{m} a_iX^i$ and $g(X) = \sum_{j=0}^{n} b_jX^j$ be two polynomials in $R[X]$ such that $f(X)g(X) = 0$. Hence, $(f(X)/1)(g(X)/1) = 0$ in $R_M[X]$ and so for each i, j, $(a_i/1)(b_j/1) = 0$ since R_M is an Armendariz ring. Therefore, there exists $s \in S(= R-M)$ such that $sa_ib_j = 0$ and this completes the proof of Theorem 2.5. □

Using Theorem 2.5 in the case when $S \subseteq U(R)$, we obtain :

Corollary 2.6. *If R is an Armendariz ring, then R_M is an Armendariz ring for each $M \in Max(R)$.*

A ring R is called a PF-ring if the principal ideals of R are flat (see [23, 24]). Recall that R is a PF-ring if and only if R_Q is a domain for every prime (resp., maximal) ideal Q of R. For example, any domain, any ring R with $wgl.dimR \leq 1$ and any semihereditary ring is a PF-ring (since a localization of a ring R with $wgl.dimR \leq 1$ (resp., semihereditary) is always locally a domain). See for instance [23, 24, 29].

Using Theorem 2.5(3), we have :

Corollary 2.7. *Any PF-ring is S-Armendariz for every multiplicatively closed subset $S := A - M$, where M is a maximal ideal of R.*

We call a ring R a P-Armendariz ring if it is $(R\backslash P)$-Armendariz, where P is a prime ideal of R. We have the following characterization.

Proposition 2.8. *Let R be a ring. Then the following statements are equivalent:*

(1) *R is Armendariz.*
(2) *R is P-Armendariz for every prime ideal P of R.*
(3) *R is M-Armendariz for every maximal ideal M of R.*

Proof. (1) ⇒ (2) Straightforward.
(2) ⇒ (3) Straightforward.
(3) ⇒ (1) Assume that R is M-Armendariz for every maximal ideal M of R and let $f(X) = \sum_{i=0}^{m} a_iX^i$ and $g(X) = \sum_{j=0}^{n} b_jX^j$ be two polynomials in R[X] such that $f(X)g(X) = 0$.

Let $i = 1, ..., m$ and $j = 1, ..., n$, we claim that $a_ib_j = 0$. Deny. Assume that $a_ib_j \neq 0$ and set $I := Ann(a_ib_j)$ which is an ideal of R such that $1 \notin I$. Hence, there exists a maximal ideal M such that $I \subseteq M$. Therefore, by (3), there exists $s \in R \setminus M$ such that $sa_ib_j = 0$ and so $s \in I(= Ann(a_ib_j)) \subseteq M$, a desired contradiction.
Hence, $a_ib_j = 0$ and so R is Armendariz. □

Now, we study the transfer of S-Armendariz property in trivial ring extension.

Let A be a ring, E be a nonzero A-module and $R := A \propto E$ be the trivial ring extension of A by E. If S is a multiplicative closed subset of R, then $S_0 = \{a \in A \mid (a, e) \in S \text{ for some } e \in E\}$ is a multiplicatively closed subset of A. In particular, $S_0 \propto 0$ and $S_0 \propto E$ are multiplicatively closed subsets of R for every multiplicative set S_0 of A.

Theorem 2.9. *Let A, E, R, S and S_0 as above. Then:*

(1) *If R is an S-Armendariz ring, then A is an S_0-Armendariz ring.*
(2) *Assume that, for each $y \in E$, there exists $s_0 \in S_0$ such that $s_0y = 0$. Then, R is an S-Armendariz ring if and only if A is an S_0-Armendariz ring.*
(3) *Assume that (A, M) be a local ring and E be an A-module such that $ME = 0$. Then, R is an S-Armendariz ring if and only if A is an S_0-Armendariz ring.*

Proof.

(1) Assume that $R(= A \propto E)$ is an S-Armendariz ring and let $f_A(X) = \sum_{i=0}^{m} a_iX^i$ and $g_A(X) = \sum_{j=0}^{n} b_jX^j$ be two polynomials in $A[X]$ such that $f_A(X)g_A(X) = 0$, where m and n are positives integers. Set $f(X) = \sum_{i=0}^{m}(a_i, 0)X^i$ and $g(X) = \sum_{j=0}^{n}(b_j, 0)X^j$ be two polynomials in $R[X]$. Clearly,

$$\begin{aligned} f(X)g(X) &= (f_A(X)g_A(X), 0) \\ &= (0, 0) \end{aligned}$$

and so for each i, j, there exists $(s_0, e) \in S$ such that $(s_0, e)(a_i, 0)(b_j, 0)(= (s_0a_ib_j, 0)) = (0, 0)$ since R is an S-Armendariz ring. Therefore, $s_0a_ib_j = 0$, as desired.

(2) Assume that, for each $y \in E$, there exists $s_0 \in S_0$ such that $s_0y = 0$. If R is an S-Armendariz ring, then A is an S_0-Armendariz ring by 1). Conversely, assume that A is an S_0-Armendariz ring and let $f(X) = \sum_{i=0}^{m}(a_i, e_i)X^i$ and $g(X) = \sum_{j=0}^{n}(b_j, f_j)X^j$ be two polynomials in $R[X]$

such that $f(X)g(X) = 0$. Set $f_A(X) = \sum_{i=0}^{m} a_i X^i$ and $g_A(X) = \sum_{j=0}^{n} b_j X^j$ to be two polynomials in $A[X]$. We have, $f_A(X)g_A(X) = 0$ (since $f(X)g(X) = 0$) and so for each i, j, there exists $s_0 \in S_0$ such that $s_0 a_i b_j = 0$ since A is an S_0-Armendariz ring. Let $e \in E$ such that $(s_0, e) \in S$. Hence,

$$\begin{aligned}(s_0, e)(a_i, e_i)(b_j, f_j) &= (s_0 a_i b_j, y)) \\ &= (0, y)\end{aligned}$$

for some $y \in E$. By hypothesis, let $s_1 \in S_0$ such that $s_1 y = 0$ and let $e' \in E$ such that $(s_1, e') \in S$. Therefore,

$$\begin{aligned}(s_1, e')(s, e)(a_i, e_i)(b_j, f_j) &= (s_1, e')(0, y) \\ &= (0, s_1 y) \\ &= (0, 0)\end{aligned}$$

as desired since $(s_1, e')(s, e) \in S$.

(3) Assume that (A, M) is a local ring and E is an A-module such that $ME = 0$. Two cases are then possible.

Case 1: $S_0 \subseteq A - M(= U(A))$.

In this case, $S \subseteq U(R)$ and the context of S-Armendariz and Armendariz coincide and we obtain the result by [6, Theorem 2.1].

Case 2: $S_0 \nsubseteq A - M(= U(A)$.

In this case, $S \cap M \neq \emptyset$ and we obtain the result by (2) since $ME = 0$ and this completes the proof of Theorem 2.9. □

Now, we can construct a class of a non-Armendariz S-Armendariz rings.

Example 2.10. *Let (A_0, M_0) be any local ring and S_0 be any multiplicatively closed subset of A_0 such that A_0 is S_0-Armendariz (for instance, take A_0 as an integral domain and so it is Armendariz). Set $A := A_0 \propto (A_0/M_0)$ as a local ring with maximal ideal $M = M_0 \propto (A_0/M_0)$ and set $R := A \propto (A/M)$ as a local ring with maximal ideal $M \propto (A/M)$. Set $S_1 := S_0 \propto 0$, which is a multiplicatively closed subset of A and set $S := S_1 \propto 0$, which is a multiplicatively closed subset of R. Then:*

(1) *R is an S-Armendariz ring.*
(2) *R is not an Armendariz ring.*

Proof.

(1) The ring A is S_1-Armendariz by Theorem 2.9 since A_0 is S_0-Armendariz. Therefore, R is S-Armendariz by Theorem 2.9 since A is S_1-Armendariz, as desired.
(2) Our aim is to show that R is not Armendariz. Let $f(X) = ((0, \overline{1}), \overline{(0, \overline{0})}) + ((0, \overline{0}), \overline{(1, \overline{0})})X$ and $g(X) = ((0, \overline{1}), \overline{(0, \overline{0})}) + ((0, \overline{0}), \overline{(-1, \overline{0})})X$ be two

polynomials in $R[X]$. We easily check that $f(X)g(X) = 0$ and $((0,\overline{1}),\overline{(0,\overline{0})})((0,\overline{0}),\overline{(-1,\overline{0})}) = ((0,\overline{0}),\overline{(0,\overline{-1})}) \neq 0_R$, as desired. $\square$

Our next theorem states necessary and sufficient conditions under which the amalgamated algebra $A \bowtie^f J$ is an S-Armendariz ring.

Throughout this result, we consistently apply the following assumptions and notations: $f : A \to B$ be a ring homomorphism, J be an ideal of B, $A \bowtie^f J$ be the amalgamation of A with B along J with respect to f, S_0 be a multiplicatively closed subset of A and we put $S := \{(s, f(s)) \mid s \in S_0\}$. Clearly, S and $f(S_0)$ are multiplicatively closed subsets of $A \bowtie^f J$ and $f(A) + J$ (and B), respectively.

Now we come to the last main result of this paper.

Theorem 2.11. *Let A, B, f, J, $A \bowtie^f J$, S_0 and S as above. Then:*

(1) *If $A \bowtie^f J$ is an S-Armendariz ring, then A is an S_0-Armendariz ring.*
(2) *If A is an S_0-Armendariz ring and $f(A)+J$ is an $f(S_0)$-Armendariz ring, then $A \bowtie^f J$ is an S-Armendariz ring.*
(3) *Assume that J is a regular ideal of B. Then $A \bowtie^f J$ is an S-Armendariz ring if and only if A is an S_0-Armendariz ring and $f(A) + J$ is an $f(S_0)$-Armendariz ring.*

Proof.

(1) Assume that $A \bowtie^f J$ is S-Armendariz and let $f_A(X) = \sum_{i=0}^{m} a_i X^j$ and $g_A(X) = \sum_{j=0}^{n} b_j X^j$ be two polynomials in $A[X]$ such that $f_A(X)g_A(X) = 0$. Set $F(X) = \sum_{i=0}^{m}(a_i, f(a_i))X^i$ and $G(X) = \sum_{j=0}^{n}(b_j, f(b_j))X^j$. Then

$$
\begin{aligned}
F(X)G(X) &= \sum_{k=0}^{m+n}\Big(\sum_{i+j=k}(a_ib_j, f(a_ib_j))\Big)X^k \\
&= \sum_{k=0}^{m+n}\Big(\sum_{i+j=k}a_ib_j, \sum_{i+j=k}f(a_ib_j)\Big)X^k \\
&= \sum_{k=0}^{m+n}\Big(\sum_{i+j=k}a_ib_j, f\big(\sum_{i+j=k}a_ib_j\big)\Big)X^k.
\end{aligned}
$$

Hence $F(X)G(X) = 0$ and so for every i, j, there exists $(s, f(s)) \in S$ such that $(s, f(s))(a_i, f(a_i))(b_j, f(b_j)) = 0$ since $A \bowtie^f J$ is S-Armendariz. Thus, $sa_ib_j = 0$ and consequently A is S_0-Armendariz.

(2) Assume that A is an S_0-Armendariz ring and $f(A)+J$ is an $f(S_0)$-Armendariz ring and let $F(X)=\sum_{i=0}^{m}(a_i, f(a_i)+j_i)X^i$ and $G(X)=\sum_{j=0}^{n}(b_j, f(b_j)+k_j)X^j$ be two polynomials in $(A\bowtie^f J)[X]$ such that $F(X)G(X)=0$. Set $f_B(X)=\sum_{i=0}^{m}(f(a_i)+j_i)X^i$, $g_B(X)=\sum_{j=0}^{n}(f(b_j)+k_j)X^j$, $f_A(X)=\sum_{i=0}^{m}a_iX^j$ and $g_A(X)=\sum_{j=0}^{n}b_jX^j$. Then $F(X)G(X)=0$ implies that $f_A(X)g_A(X)=0$ and $f_B(X)g_B(X)=0$. Hence, for each i,j, there exists $s,s'\in S_0$ such that $sa_ib_j=0$ and $f(s')(f(a_i)+j_i)(f(b_j)+k_j)=0$ since A is an S_0-Armendariz ring and $f(A)+J$ is an $f(S_0)$-Armendariz ring. Without loss of generality, we may assume that $s=s'$ since $ss'a_ib_j=0$ and $f(s)f(s')(f(a_i)+j_i)(f(b_j)+k_j)=0$. Therefore, $(s,f(s))(a_i,f(a_i)+j_i)(b_j,f(b_j)+k_j)=(0,0)$ and so $A\bowtie^f J$ is S-Armendariz.

(3) By (1) and (2), it remains to show that if $A\bowtie^f J$ is S-Armendariz, then $f(A)+J$ is an $f(S_0)$-Armendariz.

Assume that J is a regular ideal of B and $A\bowtie^f J$ is S-Armendariz. Let $f_A(X)=\sum_{i=0}^{m}(f(a_i)+j_i)X^i$ and $g_A(X)=\sum_{j=0}^{n}(f(b_j)+k_j)X^j$ be two polynomials in $(f(A)+J)[X]$ such that $f_A(X)g_A(X)=0$ and let e be a regular element of J. Set $F(X)=\sum_{i=0}^{m}(0,e(f(a_i)+j_i))X^i$ *and* $G(X)=\sum_{j=0}^{n}(0,e(f(b_j)+k_j))X^j$. Clearly

$$
\begin{aligned}
F(X)G(X) &= \sum_{k=0}^{m+n}\Big(\sum_{i+j=k}(0,e^2(f(a_i)+j_i)(f(b_j)+k_j)\Big)X^k\\
&= \sum_{k=0}^{m+n}\Big(0,e^2\sum_{i+j=k}(f(a_i)+j_i)(f(b_j)+k_j)\Big)X^k\\
&= 0.
\end{aligned}
$$

Thus, for each i,j, there exists $(s,f(s))\in S$ such that $(s,f(s))(0,e(f(a_i)+j_i))(0,e(f(b_j)+k_j))=0$ since $A\bowtie^f J$ is S-Armendariz. Hence, $f(s)e^2(f(a_i)+j_i)(f(b_j)+k_j)=0$ and so $f(s)(f(a_i)+j_i)(f(b_j)+k_j)=0$ (since e is a regular element of J), completing the proof of Theorem 2.11. $\square$

The following corollary follows immediately from Theorem 2.11, which examines the case of the amalgamated duplication and gives a complete characterization.

Corollary 2.12. *Let A be a ring, I be an ideal of A, S_0 be a multiplicatively closed subset of A, $A \bowtie I$ be the amalgamated duplication of A along I and set $S = \{(s,s) \mid s \in S_0\}$. Then, $A \bowtie I$ is an S-Armendariz ring if and only if A is an S_0-Armendariz ring.*

Now, we give a new example of a non-Armendariz S-Armendariz ring by using amalgamated duplication.

Example 2.13. *Let A be a non-Armendariz S_0-Armendariz ring, where S_0 is a multiplicatively closed subset of A, and let I be any ideal of A. Then:*

(1) *$A \bowtie I$ is an S-Armendariz ring by Corollary 2.12, where $S = \{(s,s)/s \in S_0\}$.*
(2) *$A \bowtie I$ is not an Armendariz ring since A is not an Armendariz ring (since subrings of Armendariz rings are also Armendariz).*

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Mohamed Chhiti: Laboratory of Modelling and Mathematical Structures,, Faculty of Economics and Social Sciences of Fez, University S.M. Ben Abdellah Fez, Morocco.
E-mail address: chhiti.med@hotmail.com

Salah Eddine Mahdou: Laboratory of Modelling and Mathematical Structures,, Faculty of Science and Technology of Fez, Box 2202, University S. M. Ben Abdellah Fez, Morocco.
E-mail address: salahmahdoulmtiri@gmail.com